Extremal Dynamics and the Approach to the Critical State: Experiments on a Three Dimensional Pile of Rice

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(Received 11 July 2003; published 6 February 2004)

The evolution of the growth of a ricepile is studied in three dimensions. With time, the pile approaches a critical state with a certain slope. Assuming extremal dynamics in the evolution of the pile, the way the critical state is approached is dictated by the scaling properties of the critical state itself. Experimentally, we determine the envelope of the maximal slope, which is a measure for the distance from the critical state, as well as the growth of the average avalanche size with time. These quantities obey power-law scaling, where the experimental exponents are in good agreement with those obtained from an earlier determination of the critical state properties and extremal dynamics. Furthermore, we discuss the influence of the transient state on the avalanche size distribution, which may have applications in the prevention of large avalanches in natural systems.

DOI: 10.1103/PhysRevLett.92.058702 PACS numbers: 45.70.Ht, 05.65.+b, 64.60.Ht, 89.75.Da

Self-organized criticality (SOC) was proposed more than 15 years ago to explain the ubiquitous occurrence of power-law behavior in nature [1]. However, much of the progress in the understanding of SOC theory indicates that power laws by themselves cannot be seen as a clear-cut indication of SOC [2,3]. As a matter of fact, many properties described by SOC theories cannot be studied in natural systems. Thus, well controlled experiments on model systems should be carried out in order to test the many theoretical ideas that have been put forward. From this point of view, it is surprising how few controlled experiments have been carried out testing various aspects of SOC. This may in part be due to the fact that early experiments did not find power-law behavior in the avalanches of simple, dry sandpiles, not to mention finite-size scaling of the avalanche sizes [4]. Recently, however, several systems have been shown to exhibit SOC in terms of finite-size scaling as well as power-law avalanches, such as a one dimensional pile of rice [5], a one dimensional pile of steel balls [6], magnetic vortices in superconductors [7], wet sand as a model of the formation of river beds [8], a conical pile of beads [9], as well as a three dimensional pile of rice [10]. Thus the existence of a critical point in slowly driven, nonequilibrium systems is put on a much firmer base.

A genuine understanding of the nature of SOC can, however, be gained only when the approach to the critical state is understood. This entails the self-organization process, by which the internal parameters are tuned, such that criticality is reached. There have recently been great advances from the part of theory elucidating this behavior. In the context of extremal dynamics, Paczuski *et al.* [11] have derived an equation, which they call the gap equation, describing the approach of the system to the critical state. Here the dynamics is such that extremal sites, i.e., those with maximum slope, are the points where activity takes place, such that the envelope of these maximal slopes can be described analytically. It turns out that this gap approaches a critical value as a power law, where the characteristic exponent can be obtained from the avalanche exponents in the critical state [11].

Furthermore, there is a very intimate connection of systems with extremal dynamics to the roughening of an interface [12–14]. In fact, one model which has often been used to describe the dynamics of a ricepile, the Oslo model [15], has recently been mapped exactly onto the simplest equation describing an elastic line in a random medium, the Edwards-Wilkinson equation with quenched noise [16]. In this line of reasoning is the view of Dickman *et al.* [17], who describe SOC from the point of view of absorbing state phase transitions, where the slow driving of the system leads to an attractor at the critical point. Many sandpile models can then be understood from roughening physics, which describes the movement of an elastic line in a random medium [18] and is the most natural system exhibiting an absorbing state phase transition. In this view, the reaching of the critical state is understood from the fact that active sites are removed only when their density is supercritical, while they are added only when it is subcritical [17,19]. The stationary state which will be reached is thus naturally at the critical point of the phase transition. This again leads to a description along the lines of the gap equation.

Some of the connections between roughening physics and SOC, such as scaling relations between the roughness exponents and the avalanche dimensions [11], have been observed experimentally in a pile of rice [10]. This indicates that the dynamics of such a pile is governed by extremal dynamics, which makes it an ideal system with which to study the transient behaviors and the approach to the critical state. Here we present such a study, where the evolution of a three dimensional pile of rice is observed starting well away from the critical state and getting progressively closer. In this way we can directly study a measure of the gap, which is given by the maximal local slope of the pile and hence test various scaling relations of the theory of extremal dynamics. Furthermore, we study the evolution of avalanche sizes, as well as that of the avalanche distributions, which can be used as further tests of the concepts behind extremal dynamics.

The experiments were carried out on a ricepile of a base area of $\sim 1 \times 1$ m^2. The pile is grown from a uniform line seed at one end of a half-open box, such that a geometry akin to piles in rotating-drum experiments is achieved. A detailed description of the setup can be found elsewhere [10,20]. The surface coordinates of the pile are measured by means of monocular stereoscopy, which consists of a custom-built projector and a high resolution (2048×1596 pixels) charge-coupled device camera. A set of alternating red, green, and blue lines is projected onto the pile, while an image is taken at 45° to the projection direction. The distortion of the lines then gives the surface coordinates using simple geometry [20]. The software we developed for identifying the lines leads to a resolution and accuracy of the reconstructions of 1–2 mm, which is comparable to the grain size ($2 \times 2 \times 7$ mm^3). The technique is discussed in detail elsewhere [20]. For an experimental run, the ricepile is prepared in an initial state, which is far away from the critical angle. We created a flat initial surface at an inclination of $\phi_0 \simeq 0.4$ rad. The pile is subsequently grown from a uniformly distributed line at a rate of ~ 50 grains/s. Pictures are taken every 30 s. This implies that between two images, one to two grains are added at each point along the line of growth. Thus slow driving and a separation of time scales are ensured [21]. The data discussed here come from ten different experiments, each consisting of 550 images, covering the approach to the critical state of the system.

Because of the accurate reconstructions of the surface coordinates, it is possible to characterize the local slopes of the pile, as well as the size of the avalanches between two consecutive images. The evolution of the slopes of the pile is a natural measure for the approach to the critical state. It is when the slope exceeds a critical threshold, such that the force of gravity wins over static friction, that activity starts. In the context of extremal dynamics, the places of maximal slope will be the ones activated, such that their evolution gives a measure for the approach to the critical state. In terms of the gap equation of Paczuski *et al.* [11], the envelope of the maximal slopes thus takes the role of the gap, G, which eventually reaches its critical value, f_c. Experimentally, the gap is determined via the maximum slope observed until time t, where the slopes are obtained from the average profile. The gap equation then describes the change of G with time in the transient state

$$\frac{dG}{dt} = \frac{t^{[(1-d_B/D)/(2-\tau)-1]}G}{\langle \Delta V \rangle}, \tag{1}$$

where the average avalanche size, $\langle \Delta V \rangle$, diverges as the critical state is reached, $\langle \Delta V \rangle \propto (f_c - G)^{-\gamma} \propto t$, D is the avalanche dimension, τ is the avalanche distribution exponent, and d_B is the fractal dimension of the active sites of an avalanche [11]. The asymptotic behavior of the gap can thus be easily deduced from the gap equation as $(f_c - G) \propto t^{-\delta}$, where δ is given by

$$\delta = 1 - \frac{1 - d_B/D}{2 - \tau}. \tag{2}$$

In Fig. 1, the experimentally determined gap is presented. At each time step, the local slopes of the pile were determined and their maximal value compared with the maximum of the previous gap values. If the present maximal slope exceeded the gap, it was thus increased. The main part of Fig. 1 shows a double-logarithmic plot of the difference of the gap to the critical angle, f_c. The inset shows a linear plot of the evolution of the gap for one of the experiments. As can be seen, the critical slope is approached as a power law, $(f_c - G) \propto t^{-\delta}$, over almost 2 orders of magnitude, which is indicated by the straight line in the log-log plot. We obtain an exponent of $\delta = 0.8(1)$, where the main part of the error arises from the determination of the critical slope $f_c = 0.92(1)$. Here the critical slope was determined from a direct geometrical measurement of the extent of a small sample of rice ($\sim 10^{-4} m^3$), which was slowly tilted until the critical angle was reached. Another estimation can be obtained from the maximum

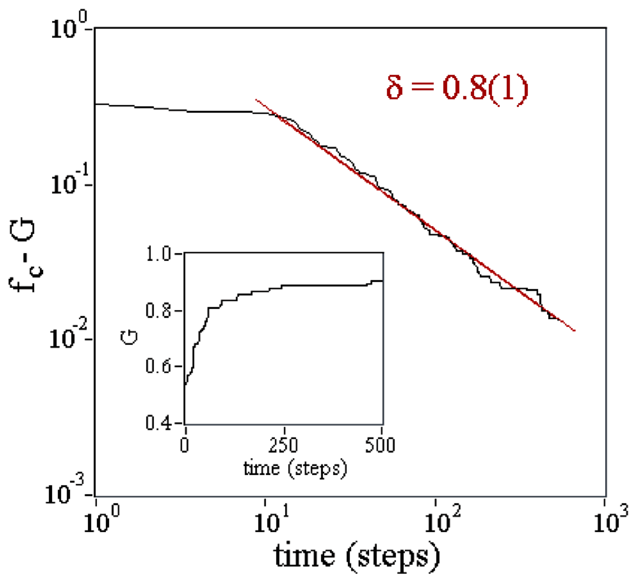

FIG. 1 (color online). Approach of the extremal slopes to the critical state averaged over all experiments. The difference of the maximum slope to the critical one approaches zero as a power law, indicated by the straight line. The characteristic exponent is given by $\delta = 0.8(1)$, where the error arises mostly from the independent determination of the critical slope. This value of δ should be compared with $\delta = 0.74(2)$ obtained from the properties of the critical state and Eq. (2). The inset shows a linear plot of the increase of the maximum slope with time for one of the experiments.

G observed over all experiments, which can be used as a lower bound and yields $f_c = 0.915$.

As discussed above, the gap equation of extremal dynamics predicts such power-law behavior, as well as the value of the exponent δ from those of the avalanche distribution exponent, τ, the avalanche dimension, D, and the fractal dimension of the active sites, d_B, in the critical state. Using the values for $\tau = 1.21(2)$, $D = 1.99(2)$, and $d_B = 1.58(3)$ determined elsewhere [10], when the system was in the critical state, one obtains a value of $\delta = 0.74(2)$, in good agreement with that found here from the evolution of the pile in the transient state.

The above derivation, however, rests on the way the average size of an avalanche grows with time in the transient regime. This can also be measured directly in our experimental system. The volume of an avalanche between two images can be obtained from the integral of their height difference over the pile area:

$$\Delta V = \frac{1}{2}\int |h(x, y, t) - h(x, y, t + \Delta t)| dxdy. \quad (3)$$

The time evolution of the avalanche size obtained in such a way is shown in Fig. 2 for one experiment.

From Fig. 2 it can be seen that there is a gradual increase in the size of the avalanches with time. However, the characteristic noise in a SOC process makes the precise dependence of the avalanche size on time difficult to discern. For this reason, in Fig. 3, we present a running average, where the avalanche sizes are averaged over a window of 100 time steps, which are subsequently averaged over all experiments. In Fig. 3 it can be clearly seen that the average size of the avalanches increases linearly with time, as would be expected from the extremal dynamics underlying the gap equation. A determination of the corresponding exponent θ, such that $\langle \Delta V \rangle \propto t^{\theta}$, yields $\theta = 1.1(1)$ in good agreement with linearity. Thus also the main feature needed for the derivation of the scaling law for the approach of the gap to its critical value can be measured experimentally.

The linear increase of the average avalanche size also has repercussions on the distribution of avalanches in the transient regime. The time evolution of the avalanche sizes as seen, for instance, in Fig. 2 has a different probability distribution than would be the case for data gathered in the critical state. As the average avalanche size increases linearly with time, smaller avalanches occur more often at early times. Thus at early times the apparent avalanche distribution exponent will be higher than in the critical state.

This is shown in Fig. 4, where the avalanche distribution exponent for avalanches in a time window of 200 steps is given as a function of time. As can be seen, the apparent avalanche distribution exponent starts at a high value and decreases with time. At long times, it reaches a value of $\tau = 1.17(5)$, which is also the exponent determined when the system is in the critical state [10].

In conclusion, we have studied the way by which a three dimensional ricepile approaches its critical state. As predicted from extremal dynamics [11], this approach is described by a gap equation for the envelope of the maximal slopes, which leads to an algebraic approach of the maximal pile slope to its critical value. The experimental value for the corresponding exponent is $\delta = 0.8(1)$, which, using the same theory, can also be predicted from the properties of the critical state to be $\delta = 0.74(2)$. These values are in good agreement, showing the applicability of extremal dynamics to the three dimensional ricepile.

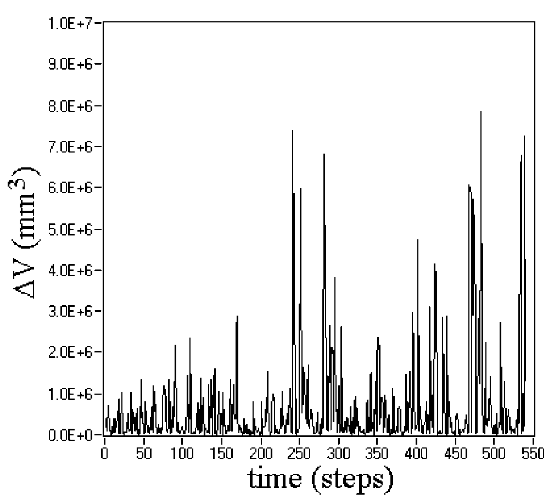

FIG. 2. The temporal evolution of the avalanche sizes in the transient regime for one of the experiments. The average size of the avalanches increases with time, as is expected from extremal dynamics. For a proper determination of the divergence of the avalanche size as the critical state is reached, it is necessary to study the average avalanche size over a short time window. This can be seen in Fig. 3.

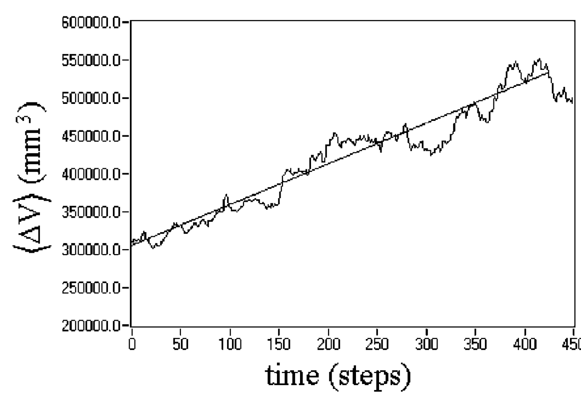

FIG. 3. A running average of the temporal evolution of the avalanche sizes in the transient regime. The averaging is carried out over a window of 100 consecutive time steps, increasing the starting time one step at the time. The average avalanche size increases linearly with time as indicated by the straight line. Assuming a power-law increase, the exponent obtained is $\theta = 1.1(1)$.

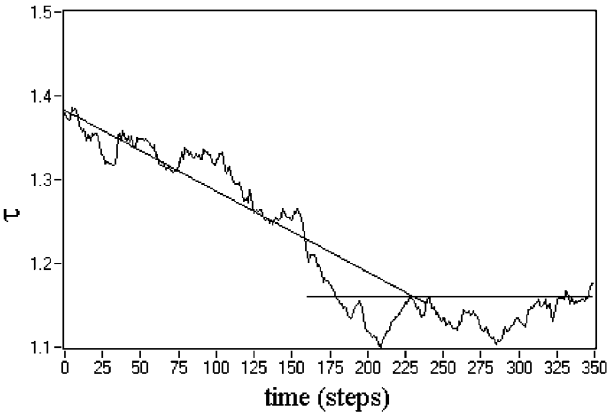

FIG. 4. The time dependence of the avalanche distribution exponent. For each point, the avalanche distribution function is determined in a time window of 200 time steps and then fit to a power-law dependence. At early times, the exponent is higher than in the critical state, whereas at late times (close to the critical state) it is consistent with our previous determination. This decrease comes from the increase of the average avalanche size with time. The lines are guides to the eye.

Furthermore, the average size of the avalanches increases linearly with time as the critical state is approached. This has been measured directly, where an exponent of $\theta = 1.1(1)$ is obtained. The same result can be inferred from the changes in the avalanche size distribution as the critical state is reached. At early times, in the transient, smaller avalanches are more likely. This leads to an increase in the apparent avalanche distribution exponent at early times. At long times the true avalanche exponent of the critical state is recovered.

This result may also have some application in natural systems, where the prevention of large avalanches is of utmost importance. It has been found recently, for instance, that the distribution of snow avalanches in parts of the Rocky Mountains remains unaffected by whether avalanches are triggered artificially or occur naturally [22]. Similarly, many of the recent forest fires in the United States were started as controlled burn-offs. As a matter of fact, the effect of more prevention leading to the occurrence of larger events even has a name in forest fire prevention: the Yellowstone effect. A more quantitative argument for this second point may lie in the fact that the distributions of fires in the western United States are the same [23], irrespective of whether one studies the period from 1150–1960 or the period from 1986–1995 in lands of the U.S. wildlife refuge. Presumably in the latter case, some care was taken in preventing big fires. In the context of SOC, this is not unexpected as the nature of the disturbance that leads to an avalanche is not important for the avalanche distribution [24]. However, as has been shown above, if human avalanche initiation were to take place in the transient period, before the critical state is reached, the avalanche distribution would indeed be different. As is hoped for in avalanche prevention, smaller avalanches are more likely to occur in that case, as is indicated by the larger avalanche distribution exponent. Naturally, such a strategy for the prevention of large events critically depends on whether it is possible to influence the system on its way to the critical state. In some systems, as in those of snow avalanches and wild fires, this seems feasible as their development lies within human time scales.

This work was supported by FOM (Stichting voor Fundamenteel Onderzoek der Materie), which is financially supported by NWO (Nederlandse Organisatie voor Wetenschappelijk Onderzoek).

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